Code Design and Performance Characterization for Code Multiplexed Imaging

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Abstract—A new sonar imaging technique which uses coded waveforms to project different sound signals in different directions has been proposed. In this context, the problem of creating a set of frequency-hopped code words that have mutually small correlation properties has arisen. In this paper we report on the design and performance characterization of frequency-hopped signals for multibeam sonar imaging. The two special cases that we consider are the fully coherent case, and the incoherent case when medium perturbations destroy the phase coherence. Two systematic design techniques are presented that are based on elements of the Galois field GF(p), where p is a prime number. One technique uses first-order Reed-Solomon code words in order to generate code words of length p - 1. The other technique is an empirical design technique which generates p - 1 code words of length p. Both techniques perform identically well under coherent processing, however, the second technique has been used to design code words that exhibit good performance bounds with respect to incoherent processing. Various theorems are proved and examples of the performance of the techniques are illustrated for time-bandwidth products of interest to the sonar realm.

INTRODUCTION

The simultaneous transmission of a number of coded waveforms can provide a substantial advantage for sonar systems that rely on active techniques for imaging. In one example, the transmission of these different waveforms in separate directions can provide increased performance in sonar imaging applications where scan rate, resolution, and sidelobe suppression is desirable. We have referred to the idea of insonifying different parts of the medium with different sounds as spatially variant insonification (SVI).

Systems that use the principle of spatially variant insonification can have superior performance characteristics over conventional systems. Typical active sonar systems rely either on pulsing and scanning a small pencil shaped beam, or on insonifying the medium with an omnidirectional pulse and beamforming on the receive. The first type of system has good sidelobe rejection and low imaging rate. The second type of system can have a high frame rate but limited sidelobe rejection. It has been demonstrated that by using SVI, higher frame rates can be achieved with superior sidelobe rejection over traditional systems [7].

The basic principle makes use of a code multiplexing scheme which is combined with spatial processing. The simultaneous transmission and reception of a multiplicity of coded waveforms makes more efficient use of the time-bandwidth properties of the medium than systems that transmit only one (possibly coded) waveform. In an SVI system, all hydrophones can simultaneously receive a superposition of the coded waveforms which have been time delayed proportionally to the distances of the insonified objects. Upon receipt, signal detection techniques afford a way of separating all of the received waveforms from each other. A substantial advantage of the coding technique is that each hydrophone is acting like N receivers where N is the total number of independent or uncorrelated signals. An extra degree of freedom over conventional systems is therefore present which can be used in a variety of ways.

In this paper we will consider the basic problem of designing N codes for a spatially variant insonification system. The desirable property of this set of codes is that they maintain their separability without sacrificing too much range resolution. It is demonstrated that the design problem is similar to that of a code division, multiaccess, spread spectrum communication system. A conceptual difference here is that each waveform is communicating information about the environment back to the interrogating system. The information contained in the reflected and coded waveform is the distance and reactivity coefficient of the objects within the insonified area. Otherwise, there are many similarities.

Since time-bandwidth (TB) considerations limit the number of degrees of freedom that the system possesses, it is interesting to compute the set of TB products that are available for underwater active sonar imaging systems. The results of a set of computations of these values is contained in Table I. Here, the available time-bandwidth products have been computed for systems of interest to high-frequency sonar imaging. The table has been determined by computing the maximum range that would lead to a 10-dB transmission loss via absorption (one way). Next, a minimum range was approximated as being 0.05 of this amount. This fixes the duration time of the sonar signal as a function of frequency since we require no
TABLE I
Representative Time-Bandwidth Products for the High Frequency Sonar Systems

<table>
<thead>
<tr>
<th>OPERATING FREQUENCY (kHz)</th>
<th>RANGE (m)</th>
<th>TIME LENGTH (ns)</th>
<th>BANDWIDTH (MHz)</th>
<th>TB PRODUCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>25</td>
<td>1.67</td>
<td>500</td>
<td>30</td>
</tr>
<tr>
<td>500</td>
<td>100</td>
<td>6.67</td>
<td>200</td>
<td>1330</td>
</tr>
<tr>
<td>100</td>
<td>400</td>
<td>26.67</td>
<td>50</td>
<td>800</td>
</tr>
<tr>
<td>50</td>
<td>1000</td>
<td>66.67</td>
<td>10</td>
<td>670</td>
</tr>
<tr>
<td>10</td>
<td>12500</td>
<td>833.34</td>
<td>1</td>
<td>830</td>
</tr>
</tbody>
</table>

* Based on 10dB absorption loss (1 way)

simultaneous reception during transmission. Some representative bandwidths at given frequencies can now be determined from existing sonar transducer manufacturers’ hardware specification [2]. A surprising result of this investigation is that the computed TB products span a somewhat narrow distribution of values which range between 500 and 1500. These values are extremely small in contrast to radar systems where TB products can range as high as $10^{-10}$.

Naturally, the use of coded waveforms for imaging involves various tradeoffs. One of these is between range resolution and intercode rejection. In an active sonar sensing system the range resolution is inversely proportional to the bandwidth of the transmitted signal. The azimuth resolution, which is dependent upon array size, is usually worse than the range resolution. If isotropic resolution in both azimuth and range is acceptable, then some degradation in range resolution may be allowed if other advantages can be obtained. In this regard, a multibeam sonar imaging system using frequency-hopped coded waveforms can trade range resolution for an increase in the number of coded sonar beams. This in turn can lead to increased mapping rates, better sidelobe rejection, or other imaging advantages.

There are many other situations in which the design techniques that we describe might be useful. One situation is a traditional one in communication sciences: when a block code is to be transmitted over a communication channel. In this case, the receiver is discriminating which of $N$ transmitted signals was received. If the transmitter and receiver are asynchronous then the situation is very similar to the one considered here.

THE SIGNAL DESIGN PROBLEM

The basic goal of the signal design problem is to prescribe a set of $k$ waveforms $s_k(t)$, $k = 1, 2, \cdots, K$ which have specific properties. In particular, it is desirable that the signals have a narrow autocorrelation function. This is because the width of the autocorrelation function is proportional to the range resolution of the system. Therefore, good range resolution is required. In addition, it is desirable that the signals can be easily separated from each other upon simultaneous receipt. For a simple correlation detector, the desirable property is that the signals have small correlation. We consider this problem initially.

Consider the continuous functions $s_1(t)$ and $s_2(t)$. The autocorrelation function of $s_1(t)$ is defined as:

$$R_{11}(\tau) = \int_{-\infty}^{\infty} s_1^*(t) s_1(t - \tau) \, dt$$

The cross-correlation function between $s_1(t)$ and $s_2(t)$ is defined as:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} s_1^*(t) s_2(t - \tau) \, dt$$

We want to design a set of $K$ signals $s_k(t)$, $k = 1, \cdots, K$, with the following properties:

$$R_{kk}(0) = \int_{-\infty}^{\infty} |s_k(t)|^2 \, dt = E \quad \text{for all } k$$

$$\frac{|R_{k\ell}(\tau)|}{E} \text{ is minimized for all } k \text{ and for all } \tau, \quad \tau \neq 0$$

$$\frac{|R_{k\ell}(\tau)|}{E} \text{ is minimized for all } \tau, \text{ all } k, \ell, \quad k \neq \ell$$

In this case [13] it is desirable that 1) each signal in the set be distinguishable from a time shifted version of itself and 2) each signal in the set be distinguishable from (a possibly time-shifted version of) every other signal in the set. For an active sonar sensing system property 1 relates to the range resolution of the system. In the SVI multibeam system, property 2 relates to the intercode rejection capabilities of the system.

In the field of spread spectrum communications, several well-established solutions exist in order to solve this problem. Frequency division multiplexing techniques could constitute a solution, however, wide-band imaging is much preferred over narrow band in order to preserve range resolution and interference rejection. Time multiplexing techniques are another alternative but, since we have uncertainty about the range of objects, the time intervals are varied.

Another set of possible solutions relies on the use of pseudorandom sequences. In a typical application [15] a single carrier frequency is phase coded as either $a + 1$ or $-1$ as $a + (\pi/2)$ or $-(\pi/2)$. One possible problem in using these codes in the ocean environment is that phase and amplitude fluctuations will disrupt them. A certain degree of perturbation is a natural consequence of transmitting these signals through the ocean [16]. In consideration of the above, phase codes are not a solution to our design problem.

In pursuit of the above goal, we have examined the feasibility of designing frequency-hop codes. These signals fulfill the design criteria that we have stated above. Most importantly, they are amenable to incoherent processing techniques. Therefore, they can be constructed and pro-
cessed so as to be somewhat “robust” with respect to the phase fluctuations that are due to oceanic variability.

**Definition of a Frequency-Hopping Signal**

A frequency-hopping signal is a frequency-coded uniform pulse train. Consider the elementary pulse of length $T$, $p_i(t)$, modulated at the frequency $f_i$:

$$p_i(t) = \begin{cases} e^{j2\pi f_i t} & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$  \hfill (6)

with $f_i = k_i / T$, where $k_i$ is a positive integer. The approximate half-power bandwidth of the signal $p_i(t)$ is given by $1/T$. Define a set of $N$ such elementary pulses $p_i(t), i = 1, \cdots, N$, such that two successive center frequencies differ by the pulse bandwidth $1/T$:

$$f_{i+1} = f_i + 1/T \quad i = 1, \cdots, N. \hfill (7)$$

The $N$ pulses cover approximately the bandwidth $N/T$.

A frequency-hopping signal $s(t)$ is defined as a uniform train of $M$ consecutive elementary pulses $12$:

$$s(t) = \sum_{m=1}^{M} p_{i_m}(t - (m - 1)T). \hfill (8)$$

The overall signal length is equal to $MT$. Assume that the $M$ distinct frequencies $f_{i_m}, m = 1, \cdots, M$, used in $s(t)$ cover the complete bandwidth $N/T$. Then the time-bandwidth product of the signal is approximately equal to $MN$. The pulse train $s(t)$ can be represented by the frequency hopping pattern $FH = (f_1, f_2, \cdots, f_M)$ where frequency $f_{i_m}$ is taken from the set of $N$ available frequencies without replacement: $i_m \in \{1, \cdots, N\}$. For further simplification the frequency-hopping pattern $FH$ can be written as: $(i_1, i_2, \cdots, i_M)$.

**Detection Algorithms for Frequency-Hopped Codes**

In considering the construction of frequency-hopped coded signals it is important to consider the detection algorithm as this will affect the code performance in a critical way. For a coherent system, the detection algorithm is easily stated as cross correlating a received waveform, a superposition of $N$ signals, with stored replicas of the individual signals. One measure of the quality of the performance of a set of code words (hereafter referred to as a code) can be determined from the intercode word cross-correlation function.

An alternate formulation of the coherent processing problem is to minimize

$$\max_{\tau, k, l} \{ R_{kl}(\tau) \}, \quad k \neq l \hfill (9)$$

The incoherent processing algorithm can be formulated as consisting of a bank of $M$ bandpass filters, followed by envelope detect and delay and sum components. The algorithm for detection of signal $s(t)$ when the received signal is $s_k(t)$ can be represented as follows. Define

$$BP_k(\tau) = \int_{-\infty}^{\infty} s_k(t) p_i(\tau - t) \, dt \hfill (10)$$

where convolving $s_k(t)$ with $p_i(t)$ is equivalent to band-pass filtering $s_k(t)$ with a filter centered at $f_i$ of width $1/T$. Now define the “incoherent” correlation between $k$ and $l$ as

$$I_{kl}(\tau) = \sum_{i=1}^{M} BP_i(\tau - L(i)T) \hfill (11)$$

where $L(i)$ generates the positions of the $f_i$th frequency in the sequence $FH$. In this case we would like to minimize again, the maximum value of this function over all pairs of code words in the code:

$$\min \max \{ I_{kl}(\tau) \}, \quad \forall \tau, k, l, k \neq l \hfill (12)$$

Certainly the problem of selecting a set of code words from among the $m!$ possible candidates that have good incoherent processing performance is extremely difficult. In comparison to the problem of generating a Costas array $[3, 4]$, in which only the autocorrelation function need be considered, in the incoherent coding case we need to consider all time delays between all of the waveforms. Furthermore, in the coherent coding case the problem is simpler in that all distinct frequency pulses are practically orthogonal. This is not true in the incoherent case in that frequency tone bursts that are close in frequency have nonzero incoherent interference with each other. This results in a more complicated metric between the frequency-hopped patterns.

There are several different strategies that can be used in order to design a set of code words with the desired properties. Exhaustive search is a technique that can and has been used when the number of elements in each code word is limited. This strategy tends to become computationally intensive very quickly as the length of the code word and the number of frequency hops increase. For our case, we have decided that even though the maximum size of the code words that we are designing may be only of length 23, the exhaustive search technique is transcomputational.

Another class of search techniques that should be considered is based on combinatorial techniques. Since the design of Costas arrays can be shown to be equivalent to filling in the squares on an extended chessboard, techniques for solving problems like the $N$ queens problem $[11]$ can be used to design frequency hop codes. Work in this area has led us to be optimistic about the possibilities of using these methods, however, we will not discuss these methods in this paper. As a final possibility, systematic generation strategies are the most appealing. Several systematic algebraic methods for constructing Costas arrays are reviewed by Golomb and Taylor $[6]$. In this paper we present two design strategies for generating frequency-hopped patterns. One strategy is based on first-order Reed–Solomon code words. Another strategy is an empirical design technique. These techniques have allowed us to generate a set of code words that has proven to possess good performance when either coherently or incoherently processed.

Given a set of $N$ available frequencies $f_i, i = 1, \cdots,
N, a frequency-division multiplexed system simultaneously transmits the N signals \( p_i(t), i = 1, \cdots, N \). We want to design a set of N frequency-hopped signals \( s_i(t), i = 1, \cdots, N \), using \( M \) different frequencies from a set of \( N \) available frequencies. By defining that each frequency appearing in FH1 is used only once, the frequency diversity of each signal is maximized. This also implies that \( M \leq N \). The number of pulses \( M \) should be as close to \( N \) as possible so that the signal bandwidth is approximately equal to the available bandwidth \( N/T \). Fig. 1 illustrates the frequency allocation strategy that is used in this technique.

Consider two frequency-hopping patterns FH1 and FH2, as described above, which correspond to two waveforms \( s_1(t) \) and \( s_2(t) \). The amplitude of the cross-correlation function between \( s_1(t) \) and \( s_2(t) \) can be minimized if the two frequency hopping patterns FH1 and FH2 are designed properly. Namely, the number of frequency matches between any pair of time-shifted patterns should be minimized. Since each frequency-hopping signal uses as much of the available bandwidth as possible, it can be assumed that any two signals from the set share all frequencies in their respective patterns. The objective of the design is to ensure that, at most, one frequency match occurs for any pair of time-shifted patterns. Such a set of patterns would be optimal and the cross-correlation amplitude level of the corresponding frequency-coded waveforms would be minimized.

For analysis purposes (use of Galois fields theory), only cyclic shifts are considered. Note that the number of matches between two circularly shifted patterns is always greater than or equal to the number of matches between two linearly shifted patterns. Consequently, we want to design \( N \) frequency-hopping patterns composed of \( M \) different frequencies taken from a set of \( N \) frequencies, such that the maximum number of matches between any pair of circularly time-shifted patterns is equal to one.

**Reed–Solomon Code Design**

A set having the properties discussed above can be constructed using Reed–Solomon (RS) code words [9], [10]. Consider \( p \) a prime number, and \( \alpha \) a primitive element of the Galois field \( \text{GF}(p) \). The field \( \text{GF}(p) \) contains \( p \) elements: \( 0, 1, \cdots, p - 1 \). All the nonzero elements of \( \text{GF}(p) \) can be expressed as \( \alpha^k \) for some integer \( k = 1, \cdots, p - 1 \) (\( \alpha^{p-1} = 1 \)). The first-order RS code words of length \( p - 1 \) are generated as follows in \( \text{GF}(p) \) [9], [10], [14]:

\[
\text{RS}(n_1, n_2) = (n_1, n_2) \left(1 \ 1 \ \cdots \ 1 \ \frac{1}{\alpha} \ \frac{1}{\alpha^2} \ \cdots \ \frac{1}{\alpha^{p-1}}\right)
\]

(13)

with \( n_1, n_2 \in \text{GF}(p) \). The total number of such code words is \( p^2 \). The code word \( \text{RS}(n_1, n_2) \) can be identified as a frequency-hopping pattern composed of \( p - 1 \) frequencies \( (M = p - 1) \). There are \( p \) available frequencies whose indices are the elements of \( \text{GF}(p) \) (let the index \( N \) correspond to the element 0). At most, one match occurs between any two first-order RS code words.

Using the analogy between code words and frequency-hopping patterns, it is easy to show that the pattern \( \text{RS}(n_1 + n_0, n_2) \) with \( n_0 \in \text{GF}(p) \) is the same as the pattern \( \text{RS}(n_1, n_2) \) circularly frequency shifted by \( n_0 \) positions. Similarly, the pattern \( \text{RS}(n_1, n_2\alpha^k) \) is the same as the pattern \( \text{RS}(n_1, n_2) \) circularly time shifted by \( k \) positions.

For our multibeam imaging system, patterns which are time-shifted versions of each other cannot correspond to simultaneously transmitted waveforms. On the other hand, assume that for standard applications of our imaging system, the possible velocities of any target are small enough to contain any Doppler shift within the bandwidth of one pulse \( 1/T \). Two patterns which are frequency-shifted versions of each other can then correspond to two different waveforms. From the properties of the RS code words, \( N = p \) frequency-hopping patterns can be generated by choosing a fixed value for \( n_2 \) for all code words and by using each element of \( \text{GF}(p) \) for \( n_1 \). Note that taking \( n_2 = 0 \) would generate single frequency pulses. The frequency-hopping patterns \( \text{FH}_i, i = 1, \cdots, N \), composed of \( M = p - 1 \) frequencies, can be generated as follows:

\[
\text{FH}_i = \text{RS}(i - 1, 1)
\]

\[
= (i - 1 \ 1 \ \cdots \ 1 \ \frac{1}{\alpha} \ \frac{1}{\alpha^2} \ \cdots \ \frac{1}{\alpha^{p-1}})
\]

\[
i = 1, \cdots, N.
\]

(14)

The resulting time-bandwidth product of \( s_i(t) \) is \( MN = (p - 1) \). This construction guarantees that, at most, one frequency match between any pair of time-shifted patterns exists.

Mersereau and Seay [8] described the general use of RS code words to generate multiple-access frequency hopping patterns. It is interesting to note that \( \text{FH}_1 = (\alpha, \alpha^2, \cdots, \alpha^{p-1}) \) is a \( (p - 1) \times (p - 1) \) Costas array generated by Welch’s rule [4], [5]. Our patterns are generated by circularly frequency shifting this Costas array in \( \text{GF}(p) \). For further reference we can denote these patterns as \( \text{RS}_p \) or \( \text{RS}_p(\alpha) = (s_1, \cdots, s_k) \) where \( \alpha \) is a primitive element of \( \text{GF}_p \) used to generate the code.

**An Alternate Design Method**

In order to propose this additional design technique, several more definitions are needed. Let \( \text{FH}_1 \) and \( \text{FH}_2 \) be
integral $M$-tuples as before

\[ \mathbf{F}_i = (i_1, i_2, \cdots, i_M), \quad \mathbf{F}_j = (j_1, j_2, \cdots, j_M). \]

(15)

Let $T_k \{ \mathbf{F}_j \}$ denote the left circular shift of $\mathbf{F}_j$ by $k$ entries so that $\mathbf{F}_j - T_k \{ \mathbf{F}_j \}$ is the difference word

\[ (i_{1+k} - j_l, i_{2+k} - j_l, \cdots, i_{M+k} - j_M) \quad \text{for} \quad k \geq 0 \]

(16)

where the subtraction may be either absolute or modulo $p$, where $p$ is a prime number corresponding to the length of the code word. Note also that the addition in the subscript is modulo $M$. Two code words $\mathbf{F}_i$ and $\mathbf{F}_j$ will be called compatible if $\mathbf{F}_i - T_k \{ \mathbf{F}_j \}$ contains only one zero element. That is, only one pair of frequencies line up for any given time delay. Define a code as being compatible if the compatibility property is true for all pairs of code words in the code. Define a distance (signed or modulo $N$) between two elements of a code word $i_k$ and $i_j$ as $d(i_k, i_j) = (k - l)$ (signed or modulo $M$).

We now define a transformation operator for nonzero elements of $\text{GF}(p)$ which we have called the “wheel of fortune” operator $\mathbf{WF}_k$. Applied to any code word of length $p$, where $p$ is a prime number, it permutes the order of the elements of the code word. The trivial transformation $\mathbf{WF}_1$ is the identity: $\mathbf{WF}_1(\mathbf{F}_i) = \mathbf{F}_i$. In general, the $k$th transformation of $\mathbf{F}_i$, $\mathbf{WF}_k$, for $0 < k < p$ is

\[ (i_{i+1}, i_{i+2}, \cdots, i_{i+p}) \]

A simple understanding is afforded by arranging the entries of the prime length code in a circle:

\[ \begin{array}{c}
   i_p \\
   i_1 \\
   \circ \circ \\
   i_2 \\
   i_s \\
   \circ \circ
\end{array} \]

The $k$th transformation starting with $i_s$ chooses the next $k$th element to be the next entry in the code word.

As an application of the above transformation, assume $\mathbf{F}_1 = (1, 2, \cdots, p)$. Then a set of code words $\mathbf{g}_k$ can be formed by applying the $\mathbf{WF}$ operator to this code word:

\[ \mathbf{F}_1 = (1, 2, \cdots, p) \]

(17)

\[ \mathbf{g}_k(k) = \mathbf{WF}_k(\mathbf{F}_1) \quad 0 \leq k \leq p - 1. \]

(18)

This construction technique will generate $p - 1$ code words of length $p$ from $\text{GF}(p)$.

It is interesting to examine the set of $\mathbf{g}_k$ code words in a little more detail. The following are the 4 elements of the 5 symbol code:

\[ \mathbf{g}_5 = [(1, 2, 3, 4, 5), (2, 4, 1, 3, 5), (3, 1, 4, 2, 5), (4, 3, 2, 1, 5)]. \]

(19)

Cyclically shifting the last two elements right by one position yields $\mathbf{g}_5 = [(1, 2, 3, 4, 5), (2, 4, 1, 3, 5), (5, 3, 1, 4, 2), (5, 4, 3, 2, 1)]$.

(20)

Note that the first element is an ascending chirp and the last element is a descending chirp (cyclical shifts of the code words preserve the compatibility). The intermediate two codes could be described as ascending and descending sawtooth waveforms. This notion generalizes to codes that consist of longer strings of elements.

We state the following properties of the $\mathbf{g}_k$ codes.

**Property 1:** Code words $(s_1, s_2, \cdots, s_m)$ composed from a set of $M$ elements with no repeated values in $s_i$, $i = 1, 2, \cdots, p$ are compatible if and only if for every pair $\alpha, \beta$ of elements of $S$ the distances between them $d(\alpha, \beta)$ are distinct among the $s_i$.

**Proof:** Elements $\alpha$ and $\beta$ of $S$ are compatible if and only if the distances between $\alpha$ and $\beta$ are equal in $s_i$ and $s_j$.

**Property 2:** Assume $p \neq 2$. $\mathbf{g}_k$ contains $(p - 1)/2$ elements which are the reverse code words of the remaining $(p - 1)/2$ code words.

**Proof:** Let $\mathbf{g}_k = (1, 2, 3, \cdots, p)$, the first element of $\mathbf{g}_k$ and $\mathbf{g}_k'_{p-1}$ be the $l$th element of $\mathbf{g}_k$. Since $p \neq 0$, we have $(p - k) \cdot l \equiv p - (k \cdot l)$ for any $0 < k, l < p$. Therefore $\mathbf{g}_k = \mathbf{g}_k_{p-k}$ and $\mathbf{WF}_{k-p}(\mathbf{g}_k)$ is the reverse code word of $\mathbf{WF}_k(\mathbf{g}_k)$.

**Property 3:** For fixed $0 < k < p$, the difference code words $(\mathbf{g}_k - \mathbf{g}_k')$ have distinct entries for $0 < i, j < p, i \neq j, 0 < k < p$, where subtraction is taken modulo $p$.\n
**Proof:** Let $0 < k < p, 0 < i, j < p - 1$, and $t = j - i$. Then the difference word under modulo $p$ subtraction is:

\[ (\mathbf{g}_k(k + 1) - \mathbf{g}_k(1), \mathbf{g}_k(k + 2) - \mathbf{g}_k(2), \cdots, \mathbf{g}_k(k + p) - \mathbf{g}_k(p)) \]

\[ = ((k + 1) \cdot i - (i + t) \cdot 1, (k + 2) \cdot i - (i + t) \cdot 2, \cdots, (k + p) \cdot i - (i + t) \cdot p) \]

Suppose that for some $0 < l, m < p, l \neq m, (k + l) \cdot i - (i + t) \cdot 1 = (k + m) \cdot i - (i + t) \cdot m$ then $(k + l) - (i + t) = (k + m) - (i + t)$ and $l = m$. But since $p$ is prime and $0 < l, m < p - k$, this implies that $l = m$.

**Property 4:** The $\mathbf{WF}$ operators $\{ \mathbf{WF}_1, \cdots, \mathbf{WF}_{p-1} \}$ when applied to any $p$ length code word $\mathbf{F}_H$ give a compatible code of maximal size $(N - 1)$. All cyclic shifts of the code words is a compatible code we consider the elements of $\mathbf{WF}_k(\mathbf{F}_H)$ under modulo $p$ distance. Let $\alpha$ and $\beta$ be elements of the code word $\mathbf{F}_H$, and let $m$ be the modulo $p$
distance between $\alpha$ and $\beta$ in $FH_n$. Then in a cyclic shift of the code word $WF_1(J_p)$, $\alpha$ and $\beta$ have (modulo $p$) distance $(m + k) \mod p$ and the numbers are distinct for $0 < k < p$, so by the property 1 and the fact that any combination of cyclic shifts of the code words $WF_1(FH)$, $WF_2(FH)$, $\cdots$, $WF_{p-1}(FH)$ gives a compatible code of maximal size.

Finally, we state without proof a property of the $WF$ operator:

**Property 5:** Let $R_p$ denote the space of all code words of length $p$. Then the wheel of fortune operators partition $R_p$.

**RESULTS OF COMPUTER SIMULATIONS**

In this section we consider the results of synthesizing two codes, $RS_{23}$ and $JS_{23}$, and applying the detection algorithms that have been previously described. Several examples of the performance of the $RS_{23}$ and $JS_{23}$ codes are described which demonstrate their properties under coherent and incoherent processing with and without medium effects. The simulations were performed on the 23 elements of $RS_{23}$ of length 22 and the 22 elements of $JS_{23}$ of length 23. In both cases, we have spaced the frequencies of the tone bursts apart by $1/T$ in order to make full use of the time-bandwidth properties of the medium. The time-bandwidth product for both systems is approximately $(23T)(23/T) = 578$. This compares favorably with time-bandwidth products that would be encountered using existing sonar systems as listed in Table 1.

Fig. 2 shows the results of applying the detection algorithm to the $RS_{23}$ codes. The plot depicts the cross correlation of a code word with all of the code words, including itself. For convenience, the autocorrelation is displayed as the most distant function in the plot. The width of the autocorrelation function in these figures at 0 delay is approximately equal to the duration of a single frequency pulse. This function is also approximately the same for any of the codes.

Fig. 3 is another representation of the data in Fig. 2. Here, the cross-correlation curves have been superimposed along the temporal axis. This affords a view of the relative heights of the peaks. As can be seen in this case, the peak-to-sidelobe level of the cross correlation and out-of-phase autocorrelation function is found to be a value $2/n$ close to the autocorrelation peak and a value $1/n$ away from the peak. Costas presents this property, which is specific to these patterns, in [4]. Thus, these frequency-hopping patterns exhibit nearly ideal correlation properties, making them suitable for a simultaneous multibeam imaging system.

In order to simulate both the ocean background noise and the receiver noise, bandwidth white noise was added to the received signal (0 dB signal-to-noise ratio). In addition, to simulate the spreading effects of the medium, a slowly fluctuating point-target model was used [16]. A Rayleigh distributed random amplitude and a uniformly distributed random phase offset was thus added to each frequency pulse. The fluctuations from pulse to pulse are therefore statistically independent.

Fig. 4 illustrates the result of using the coherent detection algorithm when spreading effects were added to the $RS_{23}$ codes. Unfortunately, the performance of the algorithm degrades substantially. Numerous large false peaks are evident in the picture. Fig. 5 depicts the result of using the incoherent processing algorithm as described above to detect the signals. As can be seen, the performance of the algorithm improves. However, numerous false peaks of large magnitude are evident. In this case, the peak-to-sidelobe level $(P/S_t)$ is only 9 dB. These results provided motivation for a search for codes which evidenced better performance with respect to incoherent processing.

An examination of the performance of the $JS_{23}$ codes demonstrates their superior performance under incoherent processing. Fig. 6 shows the result of using the incoherent detection algorithm on these codes. In this case, the
(P/Si) is approximately 16 dB. This represents a substantial reduction in the level of the sidelobes. Additional studies were undertaken in order to more completely characterize the performance of the codes. Both noise (0 dB) and spreading effects were added to the δ23 codes. The results are shown in Fig. 7. Here it can be seen that, except for the intracode correlation, the surface displays an almost ideal "thumb-tack" behavior. First, there is a fairly flat response across all codes and all delay times. In addition, the pulse compression nature of the waveforms has allowed a large amount of noise tolerance in the system.

Finally, a more realistic imaging experiment was performed as indicated in Fig. 8. Here, 3 targets have been placed in the beam patterns. Two have been located at different distances in one beam, and one in another beam. The reflected waveforms were then computed and the noise and spreading effects were simulated as above. The results of this experiment are contained in Fig. 9. Again, in this case, the system displayed an acceptable level of performance. The (P/Si) level of the three targets to the
background is approximately 15 dB. Inspection of the results of using the RS23 codes evidenced numerous large false peaks, due to the large value of the incoherent correlation.

CONCLUSIONS

In this article we have explored two design techniques in order to create a set of code words for a multibeam frequency-hopped sonar system. The desirable properties of the waveforms are that their mutual interferences be small. In the coherent signal processing case this can be equated to minimizing the intercode cross-correlation function at zero time lag. Under this processing scheme, there are many design techniques that can be used in order to minimize the value of this function. Two have been presented in this paper.

A more interesting and mathematically more difficult problem occurs when the processing algorithm being used for detection is insensitive to phase fluctuations in the medium. Here, the intercode correlation function is much more complicated as all frequency “hops” interfere with each other. This is a function of their distances in frequency space. In this case, the Reed–Solomon design technique does not perform as well as the empirical design technique. It is interesting to speculate as to why this is so.

In the Reed–Solomon coding technique, the successive codes are generated by a cyclical shift in frequency space. Thus, for zero time delay, the difference code words will all be equal to (1, 1, · · · , 1). Clearly, this is a large value for the intercode incoherent correlation function. On the other hand, the β23 codes are guaranteed to have no repeated entries for the difference code words via property 3 above. Therefore, the differences between code words are homogeneously distributed among the difference codes. This is clearly a desirable property. These properties are evident in the simulations as documented.

There are many interesting mathematical questions which can be asked about code design. The most relevant to the goals of our research are: What is the best value of peak-to-sidelobe level that can be achieved? At the current time this is an open question. In addition, in this work we have only considered cyclic shifts of the code words. If noncyclic shifts of the code words are allowed, what are the optimal codes under this detection algorithm? Experimentation with the RS17 codes by cyclically shifting and subsequent detection with noncyclic correlations has evidenced that the performance achieved by the β17 codes can be approached under similar detection algorithms. It is therefore imaginable that there is a vast number of codes which have similar performance. It may also be true that there are only a small number of codes that display optimal or near optimal performance. Additional experimentation with short length codes (length 5 was used for our experiments) has demonstrated that this is so for the one case considered.

In summary, in this paper two design strategies have been presented for synthesis of frequency-hopped codes. The basic result of this article is that SVI multibeam systems can achieve reasonable interwaveform rejection in the presence of spreading and noise effects. This has been accomplished via design techniques which have allowed the systematic generation of a set of coded waveforms. We are hopeful that future interest in this problem will allow the refinement of signal design strategies that will exploit all degrees of freedom in order to create “optimal” codes.

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REFERENCES


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